Situation Dillon2: Numerical Integration

Prompt

A Calculus class has just completed work on symbolically integrating using the basic techniques. The class is asked to evaluate $\int_0^1 \sin(x^2)\,dx$.

Commentary

Some integrals cannot be solved symbolically. There is initially some confusion when students first encounter an integral that must be evaluated by numeric methods. Starting with the chain rule for differentiating and working backwards from the integrand alleviates some of the confusion, as do examples with integration by substitution and with integration by parts. Numerical evaluation is necessary to calculate the value of this integral. Rectangular sums, the trapezoidal rule, and series are ways to estimate the value of this integral.

Mathematical Foci

Mathematical Focus 1

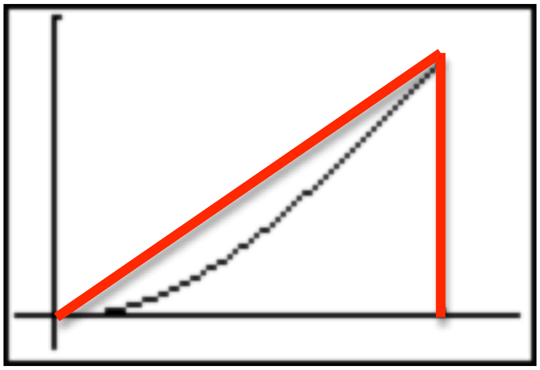
Using the chain rule for differentiation, integration by substitution and integration by parts identifies an integral as one that cannot be done by symbolic methods.

Students' first reaction to $\int \sin\left(x^2\right)dx$ is an answer of $-\cos\left(x^2\right)+C$. By checking this "answer" with the chain rule, the integrand should be $2x\sin(x^2)$. Working back and forth between various examples with the chain rule internalizes recognition of the need for the composition of functions needing the derivative of the argument in order to integrate an integrand involving composition. Since *u*-substitution is an algorithm to simplify such problems, *u*-substitution is ruled out as a method to solve $\int \sin\left(x^2\right)dx$. When integration by parts is introduced, re-introducing the prompt $\int \sin\left(x^2\right)dx$ and trying to integrate symbolically, again, does not work. The use of composition of functions and how that affects the integrand is explored in a new light. Finally, use of trigonometric substitution may be tried, but there are no suitable substitutions. Students have practiced with a variety of techniques, have learned some integrals will not succumb to any techniques, have internalized some process to recognize integrals that cannot be done symbolically, and have a need to estimate definite integrals. Further explorations with symbolic manipulators (TI-89, In-Spire, Maple) solidify the concepts and internal rule.

Mathematical Focus 2

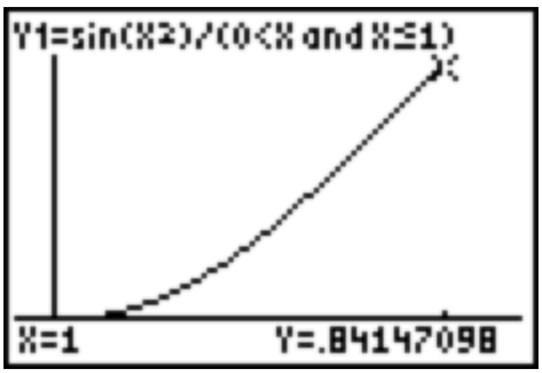
An integral may be estimated by partitioning it into polygons.

The graph indicates a single triangle is a good way to estimate the area under this curve, i.e., to estimate the value of $\int_0^1 \sin(x^2)\,dx$



The area is approximately
0.5 * 1*0.842 = 0.421

There is obviously area not needed in this approximation.

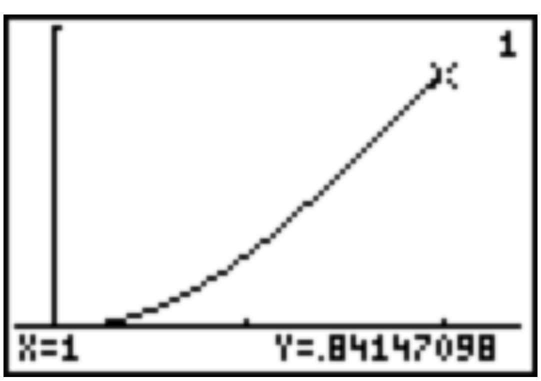


Subdividing into more shapes gives a better estimate. Since vertical lines are being used to partition along the x-axis, all the partitioning shapes will be triangles, trapezoids and rectangles. This sets the stage for further exploration.

Students may consider partitioning on the y-axis, but that goes back to the defining the integral as the area under the curve with the x-axis, since the prompt uses *dx*.

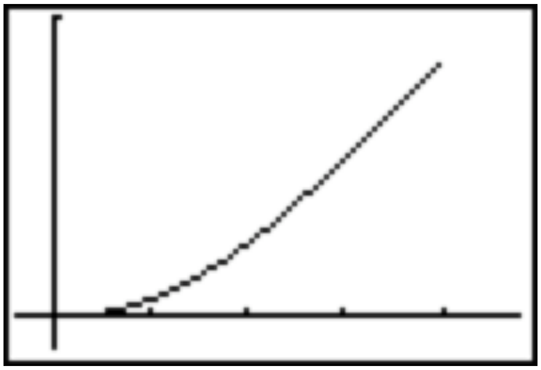
Mathematical Focus 3

Partitioning with rectangles (Riemann sums).
A common way to estimate the area under the curve is to partition the region into rectangles. One way to do that is to create *n* equal intervals on the *x*-axis, then draw rectangles whose right endpoint is on the curve.



Here, two right rectangles have been created. Obviously, this area is too large. By creating more and more rectangles, the overestimation becomes closer to the actual value. Using sigma notation and the limit of the number of partitions (n) as n approaches infinity, gives the integral value. By using different numbers of rectangles, the area may be approximated to any desired degree.

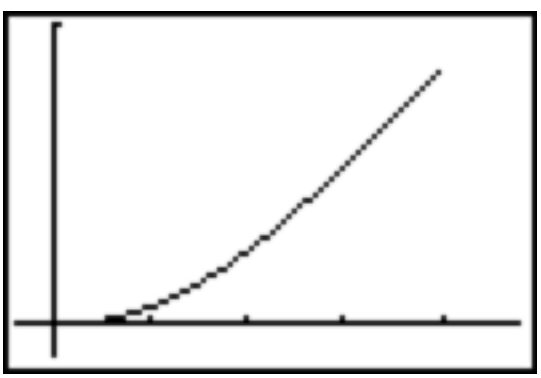
While right rectangles overestimate here, a left rectangle (left endpoint on the curve) will underestimate. Experimenting with these drawings and averaging the two results (right rectangle and left rectangle) enhances the concept of partitioning and making better estimates.



The left rectangles underestimate, and one rectangle is a "default" rectangle, as it is a line segment with area zero.

Exploration of different curves should be done so a relation between increasing, decreasing, concave up and concave down can be made with the two types of estimates (concavity is not a factor, right rectangle overestimates on increasing and underestimates on decreasing). Using horizontal lines, oblique lines and curves leads to the realization that right and left rectangles are exact when a horizontal line is the curve.

The idea of averaging the two sums is closely related to using the midpoint of the interval to create the rectangles. Comparisons between the average of the right and left rectangles should be done and generalizations made, using different curves. Exploration of different curves should be done so a relation between increasing, decreasing, concave up and concave down can be made with the estimate (Midpoint rectangles are exact with increasing and decreasing. Though it may be hard to distinguish, this estimate is over for concave down and under for concave down).



The next step is to create partitions that may have different length partitions on the x-axis. Different shaped curves may be more accurately estimated by doing this. Then, the concept is to take the limit of the largest partition length as it approaches zero (that is, the number of partitions increases to infinity) to define the more general Riemann sum.

The final polygonal estimation is the Trapezoidal Rule. Again, the relationship between the general form of the Trapezoidal Rule should be compared to the right and left rectangle approximations to see the relationship that exists (it is the average of the two). Again, experimentation determines the Trapezoidal Rule overestimates for concave up curves and is under for concave down. As with the midpoint rectangle, the estimate is exact for oblique lines.

Mathematical Focus 4

The error bounds may calculated for the right rectangle, left rectangle, midpoint rule and Trapezoidal rule.

For $I = \int_a^b f(x)dx$, the error bound rule for the right and left hand rules is the same. Let R_n be the notation for the right hand rule of *n*-subdivisions, (L_n, M_n, T_n defined in a similar manner).

For R_n and L_n, K_1 is defined as a constant that has value greater than the absolute value of the first derivative of $f(x)$ for all x in $[a, b]$. Frequently, K_1 must be estimated, but as the following rule is a set of error bounds, it is all right to overestimate K_1 (but not to underestimate it).

$\left|I - R_n\right| \le \frac{K_1(b-a)^2}{2n}$ and $\left|I - L_n\right| \le \frac{K_1(b-a)^2}{2n}$. This is an estimate (bound). The actual error may be less. Now it is possible to pick an error size (say .01) and to determine the number of intervals needed to estimate the integral to within that bound.

Similar rules exist for the midpoint rule and the Trapezoidal Rule. Though students are apt to consider the Trapezoidal Rule to be more accurate when observing the graphs, the midpoint rule has a lower error. Each of these bounds uses K_2, which is the maximum value of the second derivative of $f(x)$ for all x in $[a, b]$.

$$\left|I - M_n\right| \le \frac{k_2(b-a)^3}{24n^2} \quad \text{and } \left|I - T_n\right| \le \frac{k_2(b-a)^3}{12n^2}.$$

Mathematical Focus 5

Taylor series may be used to estimate integrals. Using Alternating Series Test or Taylor's Theorem, the error bounds for these estimates may be calculated.

Taylor series are used to write polynomial approximations of functions. The Taylor series for $\sin x$ expanded about the point $x = 0$ is

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ The series has a radius of convergence of infinity and an interval of convergence of all Real numbers, meaning the series converges for any x that is used.

Taylor series allow substitution of variables, so that $\sin(x^2)$ may easily be written as a Taylor series about $x = 0$ as

$$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{\left(x^2\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

Using the ratio test, the interval of convergence is still all Reals.
Since this is a power series, both the integral and the derivative of the series have the same radius of convergence as $\sin(x)$. Now we must integrate the series.

$$\int_0^1 \sin\left(x^2\right)dx = \int_0^1\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots\right)dx = \int_0^1\left(\sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{(2n+1)!}\right)dx$$

This gives

$$\left.\frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots\right|_0^1 = \frac{1}{3} - \frac{1}{7\cdot 3!} + \frac{1}{11\cdot 5!} - \frac{1}{15\cdot 7!} + \cdots$$

which is an alternating series. Using the alternating series test, an estimate to any desired degree of accuracy may be made. If the error is to be less than .001, the first term of the series less than .001 is $\frac{1}{11\cdot 5!}$, so only the first two terms are needed for the estimate, which is about .3095.

Post-commentary

Estimation techniques may include Simpson's Rule. Derivation of the error rules may also be explored. Ostebee and Zorn's *Calculus* is a good resource. There is far more material in the study of series and approximations, which has been assumed in the last focus.